Alternative Models of Uncertain Commodity Prices for Use with Modern Asset Pricing Methods

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This paper provides an introduction to alternative models of uncertain commodity prices. A model of commodity price movements is the engine around which any valuation methodology for commodity production projects is built, whether discounted cash flow (DCF) models or the recently developed modern asset pricing (MAP) methods. The accuracy of the valuation is in part dependent on the quality of the engine employed. This paper provides an overview of several basic commodity price models and explains the essential differences among them. We also show how futures prices can be used to discriminate among the models and to estimate better key parameters of the model chosen.

INTRODUCTION

This paper provides an introduction to alternative models of uncertain commodity prices. A model of future commodity price movements is the engine around which any valuation methodology for commodity production projects is built, whether discounted cash flow (DCF) or the modern asset pricing (MAP) methods. However, modern asset pricing methods can more readily exploit a wide array of sophisticated models of commodity price dynamics. The increasing popularity of modern asset pricing tools has therefore spurred significant research into alternative models of commodity price movements, and the analyst must understand the choices available and how accurately they capture the dynamics of actual commodity prices. In this paper we survey some of the major contributions to this research.

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The structure of the paper is as follows. First, we introduce the simple random walk model, which is equivalent to the 'permanent shock model' described in the accompanying papers by Salahor (1998), Bradley (1988), and Laughton (1998a,b,c) and we show how it is estimated using the data on crude oil prices. Second, we discuss the issue of reversion in commodity prices and why the random walk model may be an inappropriate model of prices for many commodities. We describe how futures or forward prices help to identify reversion in a commodity price series. We then provide a simple one-factor reverting model which we also estimate using the crude oil price series. The model is used in the accompanying papers by Salahor (1998) and Bradley (1998). Third, we explore the essential differences that arise from modeling commodity prices using the random walk and the reverting models. We then provide a two-factor model that combines features of the random walk and the reverting models. Fourth, we revisit the subject of futures prices, producing a more formal and detailed exposition of the information futures prices contain about the dynamics of the commodity price. Finally, we briefly discuss other, more complicated, models that have recently been developed.

THE RANDOM WALK MODEL OF COMMODITY PRICES

The Model

The simplest and most commonly used model of uncertain commodity price movements is the random walk with drift. This model writes price changes in terms of two components, an expected rate of growth or drift, μ, and a random deviation from the expected rate written as the product of a volatility parameter, σ, and an error term, u_{t+1}:

$$\frac{(P_{t+1} - P_t)}{P_t} = \mu + \sigma u_{t+1} \tag{1}$$

Because there is only one uncertain element entering into the determination of the commodity price, the random walk model is a one-factor model.

Most modern asset pricing techniques use a continuous time version of models like this. While the notation of the continuous time version may be unfamiliar to many, it can be understood first as a careful transliteration of the discrete time model given in equation (1). First, in order to analyze price changes over an arbitrarily short time interval, labeled dt, we normalize the drift and volatility of the process by the length of the time interval, writing μdt and $\sigma\sqrt{dt}\,u_{t+dt}$. The corresponding discrete time process is:

$$\frac{(P_{t+1} - P_t)}{P_t} = \mu dt + \sigma\sqrt{dt}u_{t+dt}$$

Second, we consider the limiting case in which dt approaches zero. Using the notation $dP_t = \lim_{dt \to 0}(P_{t+dt} - P_t)$, the left hand side becomes an instantaneous percentage price change: dP_t/P_t. The first term on the right hand side is unchanged. The second term, the uncertain component, is complicated: for the series of discrete variables, u_t, we substitute a new term, dz, referred to as standard Brownian Motion, where $dz = \lim_{dt \to 0} u_{t+1}\sqrt{dt}$ with dz a normal random variable. Consequently, the continuous time equivalent to equation (1) is:

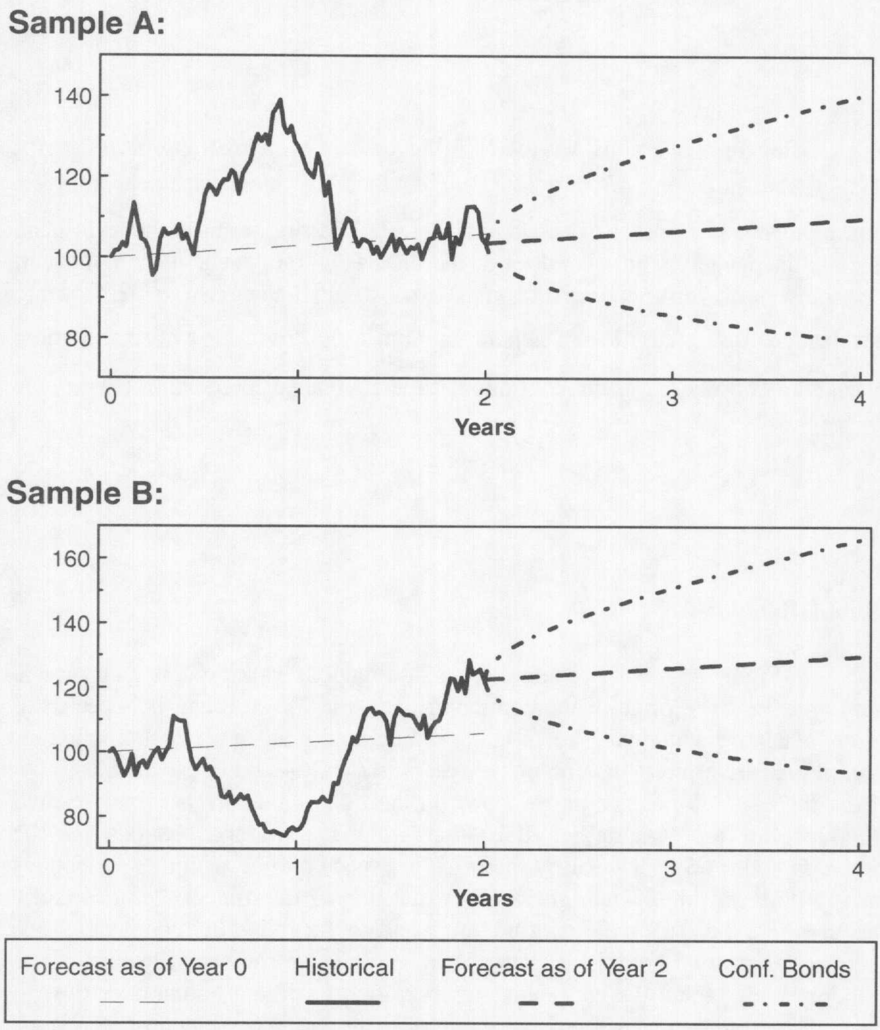
$$\frac{dP_t}{P_t} = \mu dt + \sigma dz \qquad (2)$$

Simulations

Figure 1 shows two simulations of a random walk process with $\mu=3\%$ and $\sigma=20\%$. The price of the commodity starts at $P_0=100$ and is expected to grow at 3% per annum as shown in the light dotted line. In the first simulation, sample A, the price at the end of two years is $P_2=104$ and is expected to grow from there at 3% per annum as shown in the light solid forecast line. Around this forecast are two dashed lines that are the confidence bounds for the forecast—with 66% probability the price is expected to be within these bounds at any point in time. The higher is the volatility, σ, the greater the uncertainty about where the price will be in the future and so the greater will be the distance between these two bounds at every point in time. An important feature of the random walk is that the confidence interval grows without bound over time.

In the second simulation, sample B, the price at the end of two years is $P_2=123$ and it is expected to grow from there at 3% per annum as shown in the light solid forecast line. Note that the forecasted price is based off the current price. Because the current price lies above what was forecasted at $t=0$, the current forecasted prices are all markedly greater than the forecasted prices as of $t=0$. This can be seen by comparing the trend line based off of $t=0$ against the new forecast line based off of $t=2$.

Figure 1. Random Walk: Sample Path, Forecast and Confidence Bounds



The dark solid line is the actual price of the commodity over 2 years. The initial thin dashed line is the trend or forecasted price as of time 0, starting at $100 and growing at 3% per annum. A new forecasted price is shown starting at the 2nd year price, and is equal to the price at the second year grown at 3%. The curved dashed lines are 66% confidence bounds on the forecasted price. Note that the confidence interval grow unboundedly with time. Note also that the forecasted price is revised downward or upward as the current price moves up or down. The new forecasted price is basedentirely off the current price and does not depend in any other way on the history of prices, nor on the original forecasted price.

Estimating the Parameters

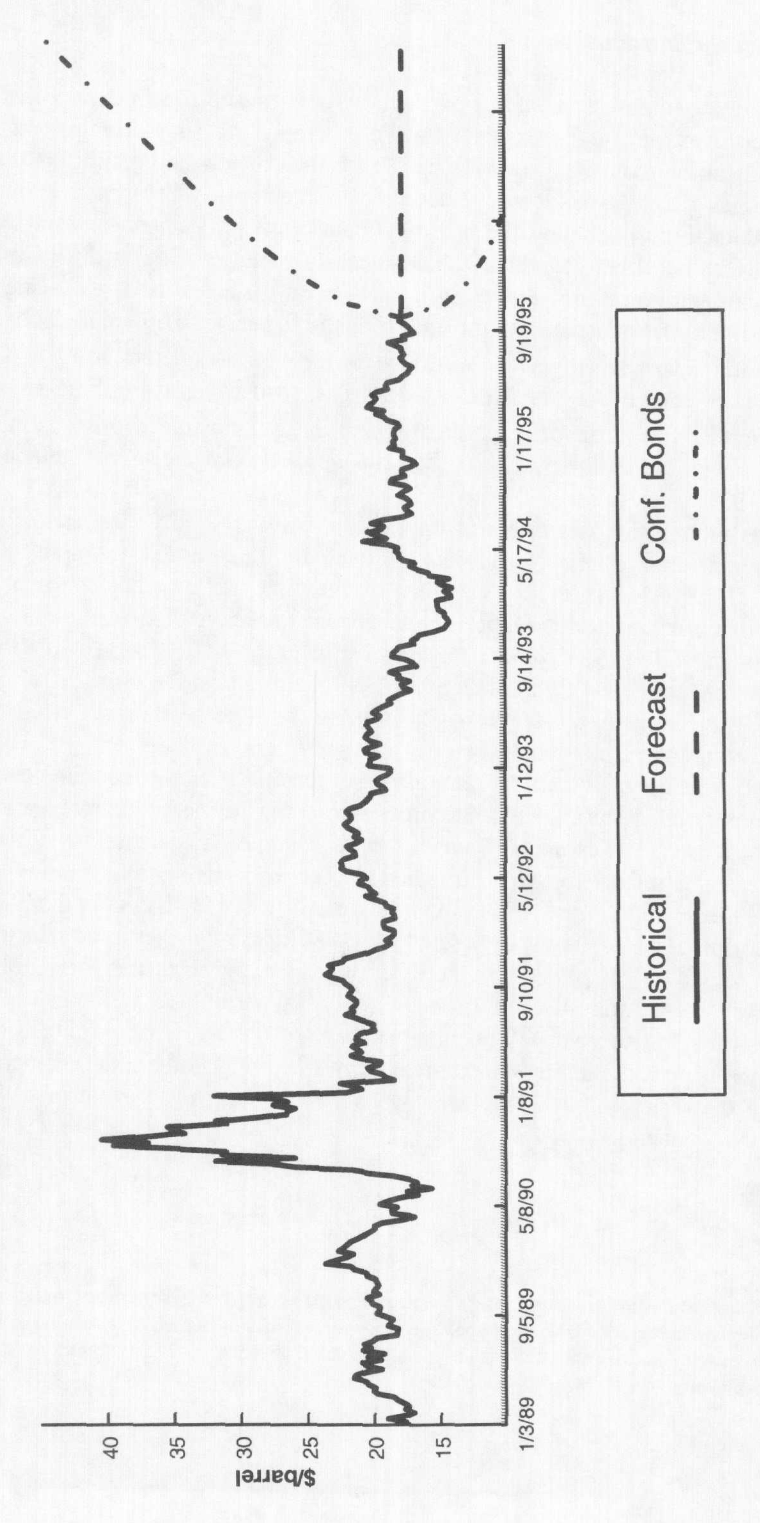
The random walk model is particularly easy to estimate from historical data. The estimator for the drift parameter is simply the annualized average price appreciation over the range of data. The estimator for the volatility is the usual estimator for the standard deviation of percent price changes over the range of data. Figure 2 shows the historical path of oil prices between January 1989 and October 1995 together with a forecast of future prices utilizing the random walk with drift model. The solid line shows the actual oil price, which declined slightly from the start to the end of this time period. This small decline translates into a very small negative estimate for the drift parameter: $\hat{\mu} \cong -1.2\%$. The forecasts shown are based on $\mu=0\%$. The dashed lines are confidence bounds for the future oil price given $\hat{\sigma}=35.1\%$, the annualized proportional standard deviation in the weekly price changes between January 1989 and October 1995.

Although the parameters of the random walk are easily estimated, the results are not always reliable and must be treated with care. For example, the estimate of drift is especially sensitive to the time period chosen for the estimation. Looking back at Figure 2, one can see that the period from January 1989 through March 1994 yields a negative estimate for drift, -4.4%, while the period from March 1994 through October 1995 yields a positive estimate, close to 10%. If the estimation were to begin or end during the peak price period of the Gulf War, the results would be even more extreme.

Estimates of volatility can also be sensitive. Both the choice of historical period and the choice of the time interval for the measurement of price changes can affect the estimate. Short intervals, such as daily or weekly, often yield greater estimates for volatility than longer intervals such as monthly.[1] Paddock, Siegel and Smith (1988) estimate a volatility of 14% based on monthly prices for imported crude oil between 1974-1980, although they note that the volatility of crude oil prices has increased significantly since the early 1980 period they were interested in for their study. According to Kemna (1993) estimates for oil price volatility may lie between 15% and 20% depending upon the historical period chosen. Gibson and Schwartz (1991) obtain an estimated volatility between 30 and 33% using weekly data covering November 1986 through November 1988.

1. Of course the estimate for volatility is predicated on the assumed model of prices. One reason that the estimate may vary, depending upon the time interval in the data used, is that the actual process driving prices does not fit the random walk model. This issue is discussed in more detail later in the paper.

Figure 2. A Random Walk Model of Oil Prices



In their work on the valuation of copper mines, Brennan and Schwartz (1985) utilized a risk-adjusted expected rate of growth in copper prices of 9% and a volatility of just over 28%.[2]

In using the random walk model for forecasting and valuation, it is important to remember that the estimates for the parameters should be forward looking. Historical values are only useful insofar as they provide the best estimates of future dynamics. There may arise situations in which the estimate derived from historical observation needs to be revised in light of additional information suggesting a shift in the true underlying drift and volatility. Analysts of commodity markets often have access to important information about future supply and demand and this information can sometimes be readily incorporated into stochastic models like the random walk. For example, suppose the analyst has information suggesting that a commodity price will fall over the near term but grow over the long term. In this case the drift parameter of the model should be made a function of time: e.g., $\mu(t)=$ -2% for $t<3$ and $\mu(t)=$ +3% for $t\geq 3,4$. Similarly, the volatility of prices need not be constant across the entire forecast horizon.

REVERSION IN COMMODITY PRICES

Equilibrium Prices in Commodity Markets

The random walk was originally popularized as a model of stock prices. Over a period of some three decades, an enormous selection of asset pricing tools have been developed on the basis of this simple model, most notably the famous Black-Scholes option pricing formula. Unfortunately, the random walk model suits commodity prices less well than it does stock prices. It is a popular model of commodity prices in large part because economists are familiar with it. And the compendium of valuation formulas that have been developed on the basis of the random walk model adds to its attraction relative to other price models for which fewer valuation formulas have been derived. Using the random walk as a model for commodity prices, however, offers familiarity at the cost of fidelity. Most commodity prices do not follow a random walk, and accurate valuations often require that the analyst go beyond this familiar model.

A central difference between stock prices and commodity prices is the predictability of price changes. If stock markets are efficient, then changes in stock prices above or below the usual rate of price appreciation are not

2. In Brennan and Schwartz (1985) the risk-adjusted expected rate of growth is the difference between the interest rate and the convenience yield which are set at 10% and 1% respectively. We will discuss this decomposition of the drift in a later section together with the distinction between the actual and the risk-adjusted rate of growth.

predictable. Each change in the stock price is independent of past changes and there should be no tendency for stock prices to return to any particular level. If stock markets are efficient, stock prices should follow a random walk with drift.[3]

For commodity prices the situation is just the opposite. It is natural to predict that a period of repeated price increases will be followed by price declines. The underlying economics of the marketplace constrains the rise or fall of a commodity's price. If the price rises significantly, then new sources of supply arise and substitutes become competitive. Conversely, if the price falls significantly and profits in the industry collapse, then capacity is reduced until profitability for the remaining producers is reestablished. These adjustments may take time, and so the commodity price can temporarily rise above or fall below its long-run marginal cost trend, subsequently returning back towards it. This process of temporary fluctuations in the short-run price and the gradual return of the commodity price to its longer-run trend is exactly the kind of dynamic that is foreclosed by the random walk model. While economic theory tells us that stock prices should not revert to any particular level, economic theory also tells us that commodity prices should revert to some long-run trend.

Tests for Reversion

To explore empirically whether a commodity price series follows a random walk or is reverting, economists have tried to identify if the time series of commodity prices is stationary. As noted above, in a random walk model the variance of the future price grows unboundedly over time, and therefore we say that the random walk is not a stationary process. A process that reverts to a long-run trend, however, is stationary. Econometricians have tests for stationarity. One group of tests, including the Dickey-Fuller test, focus on finding a "unit root" in the time-series of the commodity price.[4] Unfortunately these tests have fairly low power and unless a large number of observations are available over a long time period it is difficult to reject the null hypothesis of a random walk even when the series is generated by a reverting process. In a few cases there is sufficient data for the tests to yield results. Pindyck and Rubinfeld (1991) applied the unit root test to price series for crude oil, copper and lumber over the 117-year period from 1870 to 1986. They found that the random walk

3. Campbell, Lo and MacKinley (1997, pp.27-80) provide a more nuanced overview of the theory and evidence regarding the random walk hypothesis for stock prices.

4. For more information about unit root tests see Pindyck and Rubinfeld (1991) and Campbell, Lo and MacKinley (1997, pp.64-65). The latter text describes important ambiguities in interpreting the unit root test as discriminating between the random walk hypothesis and reversion.

hypothesis is rejected at the 5% level for crude oil and for copper, but not for lumber.

The Information Contained in Futures Prices

To this point we have focused our attention exclusively upon the 'spot' commodity price, the price for immediate delivery. Many commodities, however, have traded futures or forward contracts, and the price series for these contracts are additional sources of information about the dynamics of the underlying spot price of the commodity.[5] Even when data on spot prices does not provide clear evidence of reversion, data on futures prices often strongly supports the hypothesis that there is reversion in commodity prices.

The expectations hypothesis states that the current price of a futures contract maturing at time T equals the expectation of the spot price that will prevail at time T:

$$F_0(T) = E_0(P_T), \tag{3}$$

where F denotes a futures price, the subscript 0 denotes that it is the current quote for the futures price, and the argument T denotes that it is a price for a contract with maturity or delivery at time T, and where P denotes a spot price, the subscript T denotes that it is the price prevailing at time T, and where E is the expectations operator and the subscript 0 denotes that the expectation is taken currently. A large literature has arisen debating whether the expectations hypothesis holds or what modifications are necessary for it to do so. One important consideration left out of equation (3) is risk: the futures price should be equal to the expected spot price only after adjusting for a risk premium or discount. Later in this paper we will explore some relevant points on that matter but for now we will work with the hypothesis in the raw form described in (3). Under the expectations hypothesis, when the current spot price is above its long-run steady-state the spot price is expected to drop towards the steady-state over the coming period, and accordingly futures prices for delivery at different dates over the same period should be successively lower and pointing towards the steady state. On the other hand, when the current spot price is below its long-run steady-state, the spot price is expected to rise over the coming period, and accordingly futures prices for delivery at different dates over the same period should be successively higher and pointing towards the steady-state.

5. An important distinction must be kept in mind between futures and forward contracts. For the case in which interest rates are assumed fixed the two are identical. Otherwise they will be slightly different (Cox et al. 1981). We focus here on futures because price data is most commonly available for futures and not for forwards.

Futures Prices for Oil Reveal Reversion in the Oil Price

In Figure 3 the term structure of oil futures prices prevailing at several points in time is superimposed on the graph of the historical spot price series. Also shown is an estimate of the steady-state oil price. In general one can see that whenever the spot price of oil lies above the steady-state price, futures prices are lower, i.e., point down to the steady-state price. Conversely, whenever the spot price of oil lies below the steady-state price, futures prices are higher and closer to the steady-state price. The most extreme example comes during the Gulf War period. On October 9, 1990 the spot price of oil was near its peak at just over $40/barrel. However, on the same day the price for a futures contract for oil to be delivered three months later was $35.80 and the price for a futures contract maturing 15 months later was $26.03. The declining term structure of futures prices on October 9, 1990 reflects the expectation that spot prices would be lower in the future. Conversely, when in December of 1993 spot prices had fallen to $14.67, the price for a 3 month futures contract was $15.51 and the price for a 15 month contract was $17.49. The term structure of futures prices on each of these dates is consistent with the oil price reverting to a steady-state price as shown in the figure. They are not consistent with oil prices following a random walk.

A corollary to the principle that the futures price equals the expected spot price is the principle that the volatility of the futures price proxies the volatility in the expectation of the spot price. If the commodity price is a random walk, the volatility in the futures price should equal the volatility in the spot price. If the commodity is reverting, then the spot price will be much more volatile than the futures price. Figure 4 presents this comparison. The entire series of 15 month futures prices is overlaid on the series of spot price. Visual inspection makes transparent that the spot price is more volatile. When the spot price is high, the 15 month futures price generally lies below it. Conversely, when the spot price is low, the 15 month futures price generally lies above it.

Figure 5 displays this same effect by overlaying the term structure of futures prices for oil at 5 different dates. The short-term futures price was highly variable, moving from a high of $29/barrel to a low of just less than $15, while the longer term futures price was much less variable, moving from a high of slightly more than $22 to a low of just over $17. The higher volatility in the spot price reflects temporary shocks to supply and demand. Since these shocks dissipate over time, the futures price for longer maturity contracts reflect only the small residual effect that is expected to remain by the time of maturity. Consequently the futures price series is less volatile than the spot price series. This lower volatility for the longer maturity futures contracts is an important fact suggesting that the oil price is reverting and cannot be accurately described as a random walk.

Figure 3. Term Structure of Oil Futures Prices at Various Dates

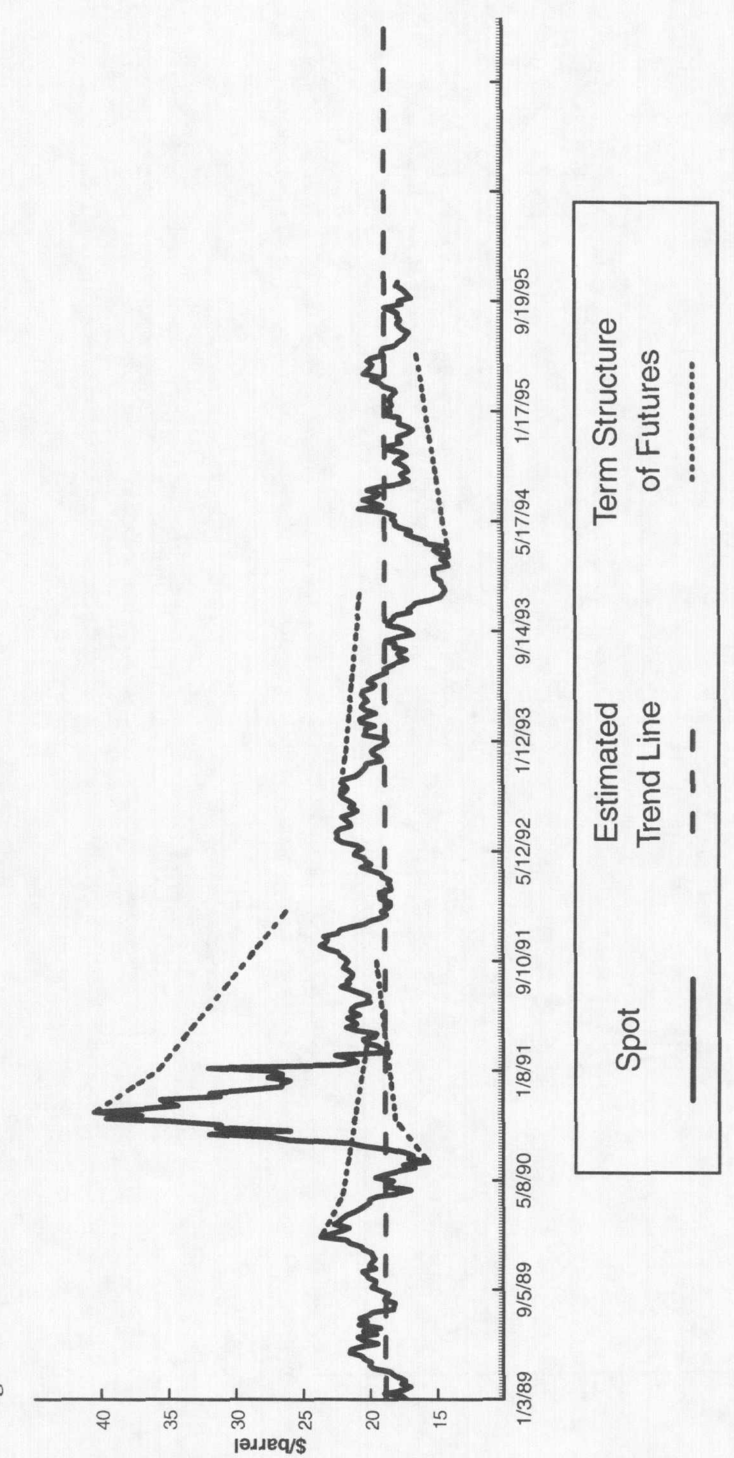


Figure 4. A Comparison of the Spot and Futures Prices Series for Oil

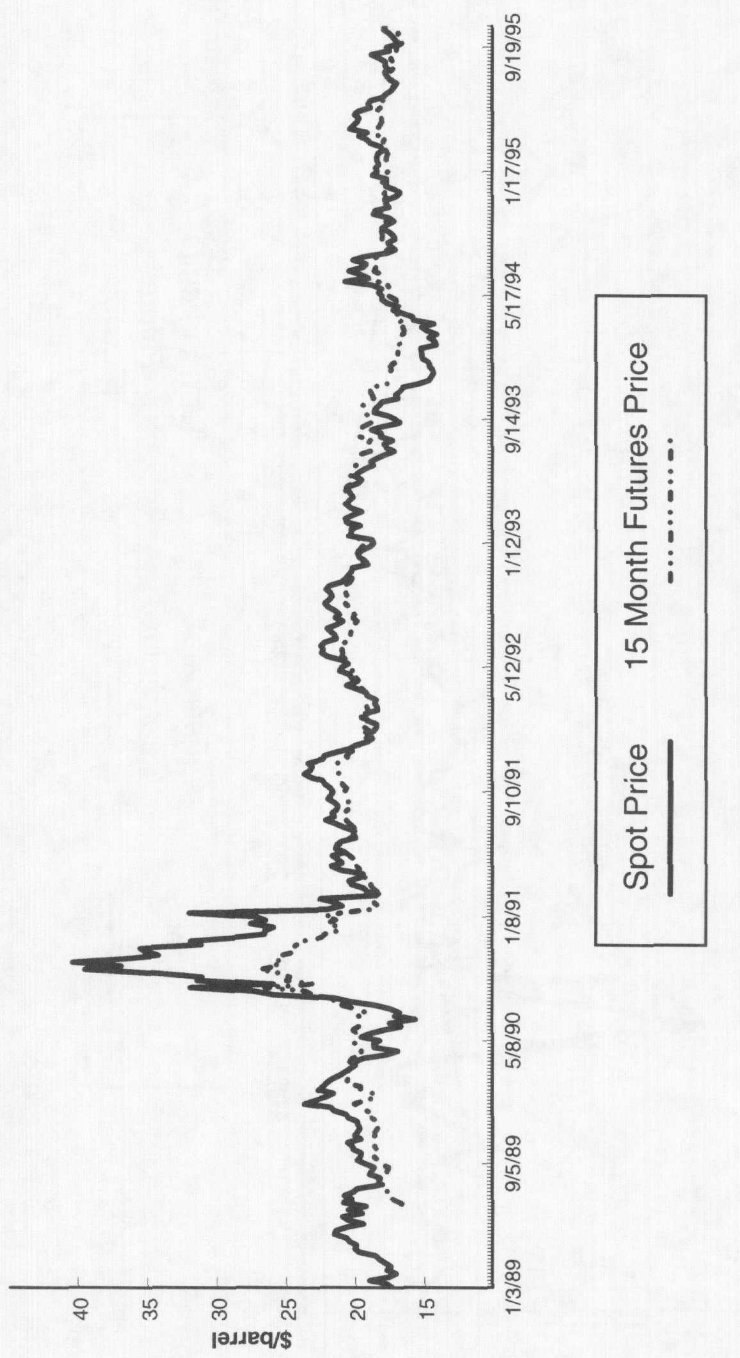
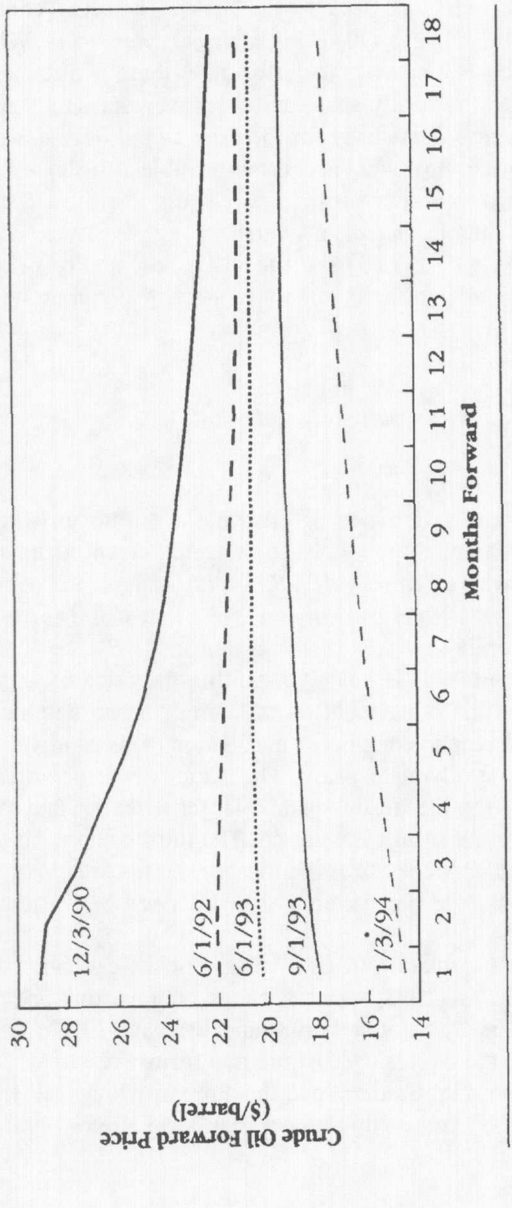


Figure 5. Volatility of the Term Structure of Crude Oil Futures Prices



*Data source: *Wall Street Journal.*

Source: Reprinted from Franklin Edwards and Michael Canter, "The collapse of Metallgesellschaft: unhedgeable risks, poor hedging strategy, or just bad luck," *Journal of Applied Corporate Finance*, Spring, 1995, Figure 3.

A Reverting Model of Commodity Prices with One-Factor

So far we have spoken of reversion without being very precise about what was meant. One simple model of a reverting commodity price distinguishes between the steady-state or long-run price of the commodity and the current price. The steady-state price, P_t^*, grows at a predictable rate, μ, yielding a well defined trajectory for the steady-state price: $P_t^* = P_0^* e^{\mu t}$. The actual commodity price, P_t, bounces around this steady-state, but always reverts to it. Whenever the actual commodity price lies below the steady-state, there is a tendency for the price to rise back up to the steady-state, and whenever the actual price lies above the steady-state, there is a tendency for the price to fall back down to the steady-state. The actual price, however, is constantly subject to shocks, so that at any given moment it may move further away from the steady-state, or approach it more quickly than expected, or overshoot it.

What we have just described is a model of commodity prices that decomposes all price changes into three components and that can be written in continuous time notation as:

$$\frac{dP_t}{P_t} = \mu dt + \kappa \ln\left[\frac{P_t^*}{P_t}\right] dt + \sigma dz \tag{4}$$

The first component is the rate of growth, μ, due to growth in the steady-state price of the commodity, P_t^*. Note that the percent change in the steady-state price is everywhere given by $dP_t^*/P_t^* = \mu dt$. The second component is the growth or decline that is due to reversion. Note that if the current price is above the steady-state price—i.e., $P_t > P_t^*$ so that $\ln(P_t^*/P_t) > 0$, then the second component in equation (4) is negative, pulling the price back down to steady-state price, P_t^*. On the other hand, if the current price lies below the steady-state price, then the second component in equation (4) is positive, pulling the price back up to the steady-state price. The parameter κ determines the severity or rate of reversion to the steady-state. A larger κ implies that the price pulls more quickly back to the steady-state price. The third component of price changes is the uncertain component and is identical to the uncertain component in the random walk so that the parameter σ still describes the instantaneous volatility of prices.

The model given in equation (4) is a version of the exponential decay model in Laughton and Jacoby (1993) which is called the reverting price model in the accompanying papers by Salahor (1998) and Bradley (1998). Laughton and Jacoby (1993) outline the equivalence of the two forms, one of which uses price forecasts as do Salahor and Bradley, and the other of which uses the price itself, as is done here. A version of this model is used in Ross (1995) and in

Schwartz (1997).[6] Since equation (4) has only one uncertain variable, dz, it is a one-factor reverting model. Later we discuss some two- and three-factor reverting models.

Simulations

Figure 6 shows two simulations of this one-factor reverting process with $\mu=3\%$ and $\sigma=20\%$ as in the earlier simulation of a random walk, and with $\kappa=2$ and $P_0^*=100$. The light dotted line traces this steady-state price growing over time at the rate of 3%, $P_t^* = P_0^* e^{\mu t} = 100e^{0.03t}$. This is the steady-state to which the price reverts. In the first simulation, sample A, the price at the end of two years is $P_2=104$, slightly below the steady-state price of $P_2^*=106$, and therefore is expected to rise initially at a rate greater than 3% until it returns to the steady-state path. This forecasted path is shown in the light solid line projecting past $t=2$. Around this forecast line are two dashed lines that are the confidence bounds for the forecast. In contrast to the random walk, the confidence interval grows only to a limit and then remains approximately constant for the remaining time.

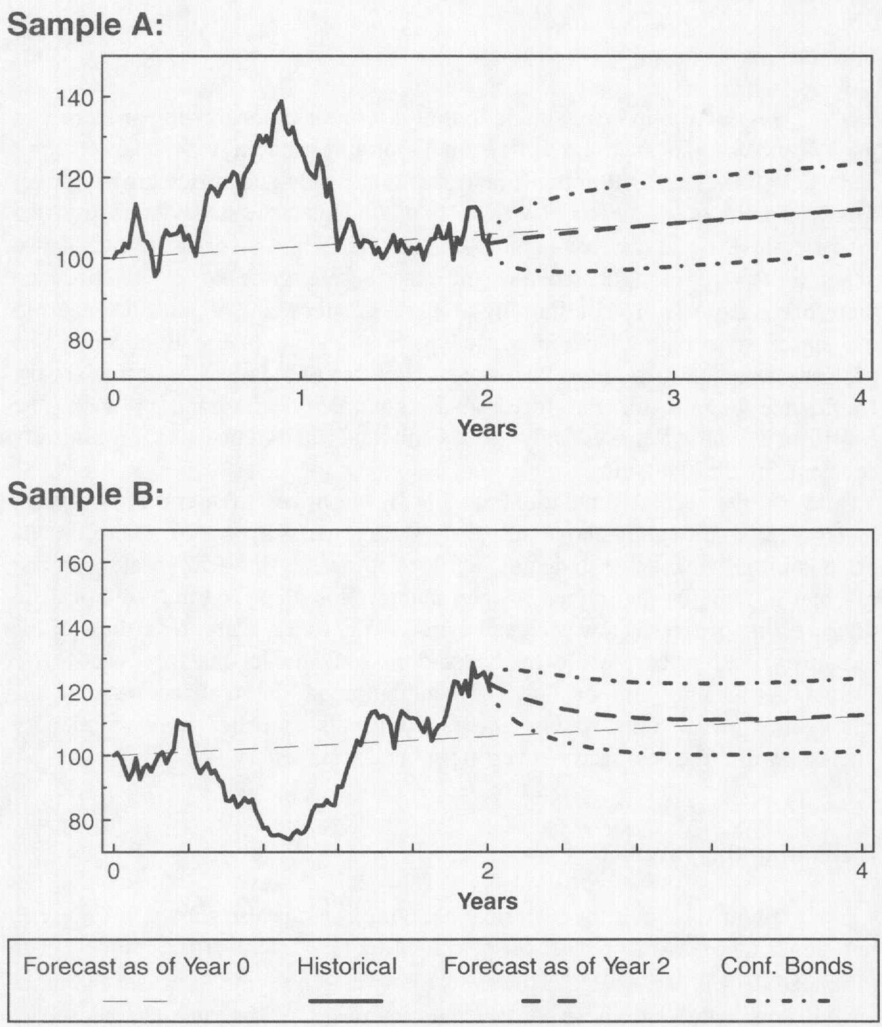
In the second simulation, sample B, the price at the end of two years is $P_2=123$, significantly above the steady-state price of $P_2^*=106$. Therefore the price is initially expected to decline until it approaches the steady-state path and resumes growing at the rate of 3% per annum. Note that the forecasted price in the reverting process is always located around the steady-state, regardless of the past history of prices, while in the random walk model the forecasted price shifted permanently up or down depending upon the realized path of the commodity price. This is why the random walk is called a non-stationary process, while the one-factor reverting model is stationary.

Estimating the Parameters

Figure 7 shows the historical path of oil prices also shown in Figure 1, but superimposes a forecast of prices utilizing the reverting model. The estimated drift is $\hat{\mu}=-5.1\%$. However, in order to keep the forecasts and other analyses comparable to those constructed previously for the random walk model shown in Figure 1 we have imposed the assumption that $\mu=0\%$ as we did there. The estimated volatility is $\hat{\sigma}=34.4\%$ and the estimated steady-state price is $\hat{P}_0^*=\$18.86$/bbl. The solid line shows the forecasted oil price over time,

6. By approximately changing the variable, equation (4) can be rewritten to describe changes in the log of the price as a case of the Ohrenstein-Uhlenbeck process.

Figure 6. A Reverting Model: Sample Path, Forecast & Confidence Bounds



The dark solid line is the actual price of the commodity over 2 years. The dashed line is the steady-state price, starting at $100 and growing at 3% per annum. A forecasted price is shown starting at the 2nd year price. The forecasted price moves from the current price towards the steady-state price. The curved dashed lines are 66% confidence bounds on the forecasted price. Note that the confidence interval is bounded: at distant horizons the interval approaches a constant size. Note also that only the short-term forecasted price is revised downward or upward as the current price moves up or down. In the long-term the forecasted price always approaches the steady-state.

rising slightly from \$17.84/bbl in October 1995 back to the steady-state price of \$18.86/bbl. This is to be contrasted with the random walk model in which the forecasted prices are constant at the October 1995 price of \$17.84/bbl. Despite the fact that the drift parameter used in the forecasts is the same in the two cases, the forecasted price path is different. Notice also that the confidence bounds on future oil prices are significantly tighter for the reverting model, this despite the fact that the estimated volatility parameter is virtually identical to that used in the random walk model.

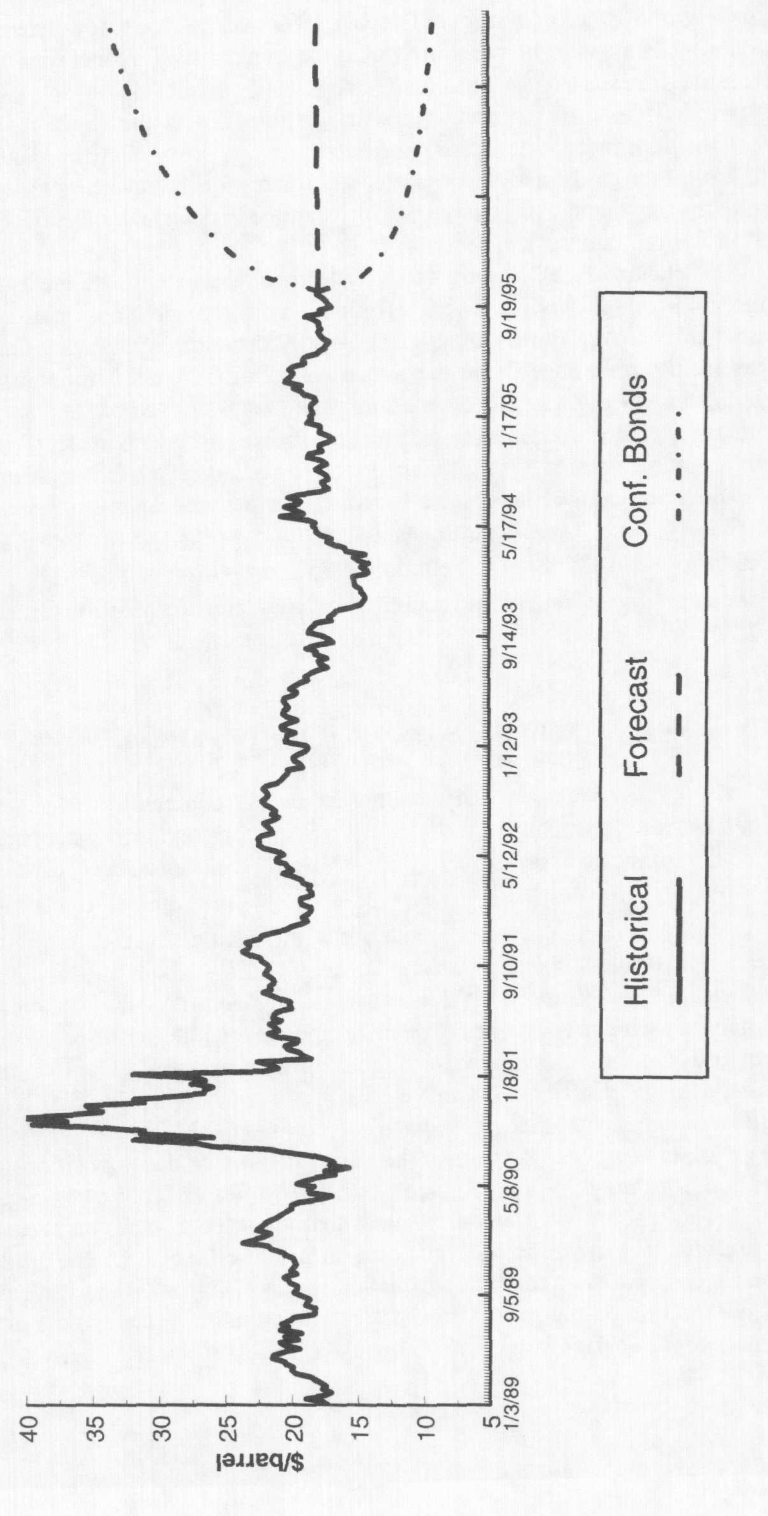
Schwartz (1997) uses a version of this model to estimate the dynamics of the gold, oil and copper prices. He finds that the model could not be fitted to gold prices over the time period 1985-1995, which is evidence that gold prices are not reverting. Of all the traded commodities, gold is the most like a pure financial asset for which the random walk model should hold and for which the theoretical presumption for a reverting process is the weakest. Thus this result is not surprising. Using oil prices from January 1985 through February 1995 and setting $\mu=0$, he obtains estimates for the rate of reversion ranging from $\hat{\kappa} =$ 03 to 0.7 and volatility between $\hat{\sigma} =$ 26% and 33%. Using copper prices from July 1988 through February 1995 and again setting $\mu=0$, he obtains an estimate for the rate of reversion of $\hat{\kappa} = 0.4$ and a volatility estimate of $\hat{\sigma} = 23\%$.

Interpreting Historical Price Series Using the Two Models

Which model the analyst chooses for the commodity price process determines how the analyst 'reads' the data supplied by the past history of commodity prices. For example, for the oil data described earlier the unconstrained estimates for drift were $\hat{\mu} = -1.2\%$ for the random walk model and $\hat{\mu} = -5.1\%$ for the reverting model. The different estimates are a result of the fact that the two models 'assume' different things about any given price movement. At the simplest level one can simply note that the random walk decomposes each price change into two components while the reverting model decomposes the same price movements into three components. The two models are bound to yield different estimates for the parameters of the drift and the volatility components that they have in common. Of course this simple observation is just a starting point. The nature of the two decompositions helps to explain how and why the parameter estimates differ under the two models.

The random walk model assumes that the price changes on successive days are identical in structure, containing in part the usual drift and in part a stochastic element. Consequently the movement every day is taken as an equally clean observation on the drift. If one day is a large fall in price and the other a modest rise, the analyst using the random walk model estimates an average price

Figure 7. A Reverting Model of Oil Prices



change that is slightly negative. On the other hand, the one-factor reverting model assumes that a portion of the movement each day is a reversion towards the steady-state. The analyst using the reverting model first extracts this portion of each day's movement before estimating the drift. If a portion of the second day's rise in price is considered a reversion to the steady-state while little if any of the first day's drop is considered as reversion, then the analyst using the reverting model may end up estimating a negative drift on average.

The same difference in how the data is read has an impact on the analyst's estimate of the price volatility. The analyst using the random walk model treats one portion of each day's price movement as a result of the drift, and the remaining portion as an independent observation of the average size of daily price swings or the average volatility. The analyst using the reverting model also treats one portion of each day's price movement as a result of the drift, differently estimated, however. The analyst then treats a second portion of each day's price movement as a reversion to the steady-state price. Only after subtracting these two pieces of the price movement does the analysts now calculate the daily volatility using the remaining 'unexplained' portion of the price movement that day. Both because the daily drift is estimated differently, and because the one model incorporates a reversion to the steady-state, the two models yield different estimates of the daily volatility. This is why the estimate for oil price volatility is marginally different under the two models, despite the fact that the same data series is used in both cases.

Reversion is also the likely explanation for the fact noted earlier that the estimate for volatility in a random walk model is sensitive to the interval over which price changes are measured. If the process is actually reverting, variation in monthly prices is generally smaller than variation in daily price changes since a portion of the daily price changes will be cancelled out over the month as the price moves back towards the steady-state. Estimating the parameters of the reverting model is more complicated than for the random walk. A detailed discussion of the problem is given in Lo and Wang (1995) and in Campbell, Lo and MacKinley (1997).

Permanent and Transitory Shocks

Every movement in a commodity price can be broken down into a permanent and a transitory component. If the price of oil climbs 25% over six months, an oil producer is going to want to know whether the new price level is likely to last for any significant period of time or if prices are more likely to return to their previous level. How much of this price rise is caused by a change in the underlying economics of exploration and development, and how much is caused by temporary aberrations in supply or demand? The random walk and the reverting processes provide very different answers because the two models 'read' the data differently.

The random walk model *assumes* that every price movement is a permanent change to the long-run price trajectory, while the one-factor reverting model assumes that every price movement is just a temporary deviation from the unchanged long-run price trajectory. Under the random walk model, a quick run-up in the price of oil is treated as permanent and leads to a corresponding upward revision in the forecasted path of oil prices. In the case of the one-factor reverting model the run-up in price is treated as temporary. The long-run path of prices remains unchanged although in the short-run the forecasted path is shifted upwards and tilted down as the price reverts to the long-run path.

Of course, it hardly seems reasonable that one should have to choose between a model that assumes all price changes are permanent and a model that assumes all price changes are temporary. For most commodities price changes are sometimes the result of permanent shocks to the long-run path of prices and other times the result of temporary disturbances, and most of the time price changes are a result of both factors. A realistic and useful model of commodity prices should incorporate this fact.

A Two-Factor Reverting Model of Commodity Prices

A model can be constructed which captures these two forces simultaneously. In this model the current commodity price varies around the steady-state, but over the long-run the steady-state price itself is stochastic. The current commodity price reverts to the steady state target, although the target is moving. The model is described by three equations:

$$\frac{dP_t^*}{P_t^*} = \mu_{P*}\,dt + \sigma_{P*}\,dz_{P*} \tag{5}$$

$$\frac{dX_t}{X_t} = \kappa \ln X_t dt + \sigma_X dz_X \tag{6}$$

$$dz_{P*}\,dz_X = \rho_{P*X}\,dt \tag{7}$$

where P_t^* is the steady state price and X_t is the ratio of the current spot price to the steady state price, $X_t = P_t^* / P_t$. The drift in the steady state price is μ_{P*}. The terms dz_{P^*} and dz_X, are the two distinct uncertain factors determining the current commodity price. The former represents shocks to the steady state price—e.g., shocks to the long-run cost structure of producers, technological innovation and so on—while the latter represents shocks to the current commodity price—e.g., transitory increases or decreases in immediate supply and demand. The parameter ρ_{P^*X} is the correlation between these two uncertain parameters. The parameter σ_{P^*} is the volatility of the steady state price and the parameter σ_X is the volatility of the ratio of the current price to the steady-state price.[7]

In this model uncertainty about distant forecasts for the price of a commodity is determined primarily by the volatility in the steady-state price, σ_{P^*}. Volatility in the ratio of prices, σ_X, causes temporary deviations from the steady-state price, but adds little lasting uncertainty about distant forecasts. Notice that the model does not actually contain a parameter for the volatility of the observed spot price, σ_P. The volatility in the spot price is a product of both the volatility in the steady-state price and the volatility in the ratio, σ_{P^*} and σ_X .

As with the one-factor model, estimators for the parameters of two-factor reverting models are complicated. A formal treatment is given in Lo and Wang (1995) and in Campbell, Lo and MacKinley (1997) and in briefer form in Schwartz (1997).

A FORMAL MODEL OF THE TERM STRUCTURE OF FUTURES PRICES

We described in an earlier section that futures prices are an important additional source of information with which to identify reversion in the commodity price. In order to fully exploit the data series of futures prices in estimating a model of the commodity price it is important to formally describe how futures prices at different maturities are related to the current spot price. The expectations hypothesis mentioned earlier is a starting point, but it relates the futures price to the expectation of a later spot price instead of to the current spot price. How are the current futures prices and the current spot price related to one another?

Futures Prices in the Certainty Case

To answer this question we start with a Hotelling-style model in which the commodity price is not stochastic and follows a known path growing at the rate $\mu(t)$. This growth rate can be broken down into two components, $\mu(t) = r(t) - \delta(t)$, where $r(t)$ is the instantaneous interest rate at time t and δ is the marginal

7. A version of this model appears in Schwartz (1997). The version used here is thanks to Ramzi Zein.

net convenience yield at time t, or the convenience yield, for short. The interest rate represents the normal riskless rate of appreciation required on an asset over an instant in time. The convenience yield is the aggregate shortfall between the actual rate of growth and the riskless rate of interest. It captures all of the factors that cause the rate of appreciation of the price to fall short of the interest rate. For example, inventory of a commodity helps producers avoid stockouts and keep production operating smoothly.[8] Because the producer earns this implicit yield on the stock of the commodity, it can profitably hold the inventory even if the price grows at a rate that is below the interest rate by the amount of this convenience yield. On the other hand, since the inventory imposes storage costs on the producer, this pushes the required rate of appreciation higher. The parameter δ captures the net effect of these and any other factors causing the growth rate to fall short of the interest rate—hence the term net convenience yield.

Assuming that the rate of growth, μ, the interest rate, r, and therefore the convenience yield, δ, are each constant over time, the current futures price for a contract with maturity at date T, $F_t(T)$, is related to the current spot price, P_t, by the following equation:

$$F_t(T) = E_t(P_T) = P_t e^{\mu(T-t)} = P_t e^{(r-\delta)(T-t)} \tag{8}$$

The equation says that the futures price is exactly equal to the expected spot price at T, which in turn is equal to the current spot price grown at the rate r-δ.

Futures Prices in the Random Walk Model

When we introduced the expectations hypothesis earlier we noted that we were abstracting from the question of whether the futures price incorporated a risk premium or discount. Equation (3) stated that the futures price was equal to the expected spot price, pure and simple. In the certainty case, as shown in equation (8), this simple version naturally holds since there is no risk. However, in any model with uncertainty we must account for the possibility that the futures price will incorporate a risk premium or discount. In the random walk model it can be shown that the futures price is related to the spot price by a variation on equation (8):

8. McDonald and Siegel (1984), Brennan (1991), Pindyck (1994), and Litzenberger and Rabinowitz (1995) all examine various aspects of the concept of the 'rate of return shortfall' or convenience yield.

$$F_t(T) = E_t(P_T)e^{-\lambda(T-t)} = P_t e^{(\mu-\lambda)(T-t)} = \bar{E}_t(P_T) = P_t e^{(r-\delta)(T-t)} \tag{9}$$

where λ is the market price of risk and can be positive or negative depending upon whether there is a risk premium or discount, and where $\bar{E}_t$ is the risk-adjusted distribution of the spot price.[9] Note that in the random walk model $\mu \neq r\text{-}\delta$ as in the certainty case, but rather $\mu\text{-}\lambda = r\text{-}\delta$.

From equation (9) we can see that futures data allow us to jointly estimate the drift in the commodity price and the market price of risk, μ and λ, using the interest rate and the convenience yield. To isolate an estimate of the drift itself requires an estimate of the market price of risk. Since estimates of the market price of risk are very unstable, this defines the limit to which data on futures prices adds to the information about the drift available from the spot price series itself.

In many cases the problem of estimating the drift and the market price of risk can be finessed. For certain problems the joint estimate of drift and the market price of risk is a sufficient statistic for valuation, and so we need not separate out the drift and the market price of risk. The intuition behind this concept can be captured in the fact that the futures price is by definition the current value of a delivery of the commodity at the contract maturity date. Many combinations of drift in the spot price and the discount for risk are consistent with the same futures price—a higher drift is cancelled by a greater discount for risk. But all that matters for the valuation of an anticipated delivery of the commodity is the futures price, regardless of the particular combination of drift and discount for risk. Another way of putting the same point is to say that all that matters for many valuations is the expected spot price under the risk-adjusted distribution, $\bar{E}_t$. Futures prices enable us to estimate the parameters of the risk-adjusted distribution—in this case the proportional convenience yield, δ—even when they do not significantly improve our estimates of all of the parameters of the underlying distribution or process driving the commodity price, i.e., μ.

Futures Prices in the Reverting Models

The term structure of futures prices is more complicated in the case of the reversion models. In the one-factor model the term structure is given by:

9. See Ross (1978).

$$F_t(T) = \bar{E}_t(P_T) = \exp\left[e^{-\kappa(T-t)} \ln P_t + (1 - e^{-\kappa(T-t)})\alpha^* + \frac{\sigma^2}{4\kappa}(1 - e^{-2\kappa(T-t)}) \right] \tag{10}$$

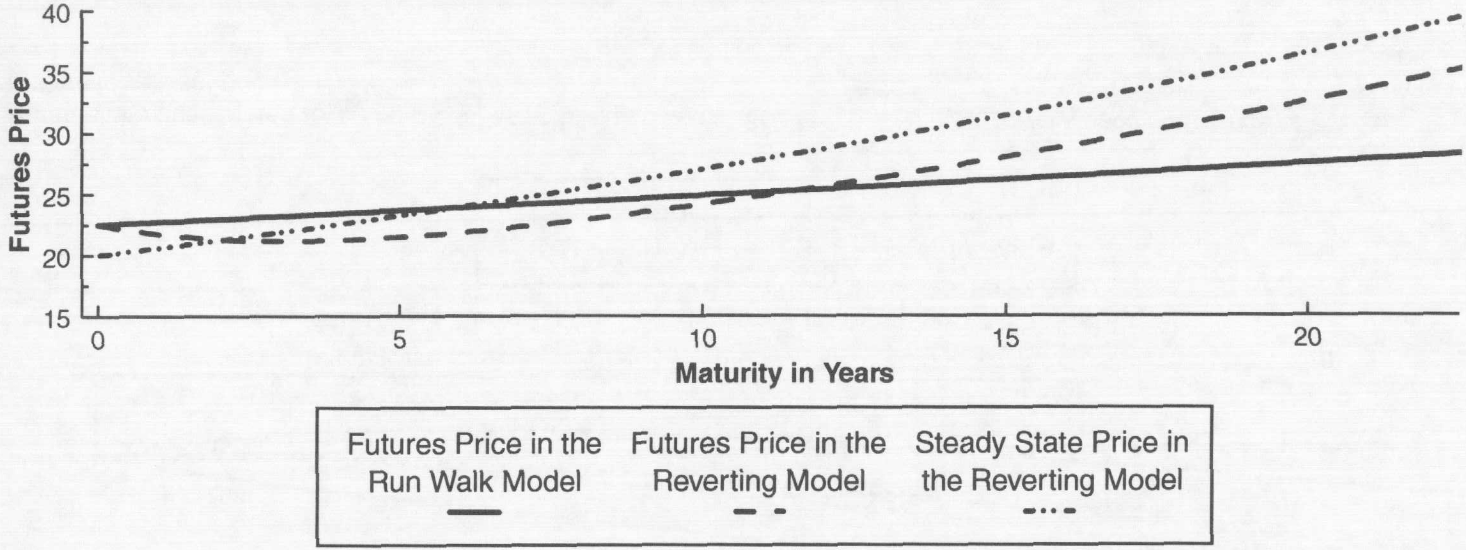
where $\alpha^* = \ln P_0^* + \mu - (\sigma^2/2\kappa) - \lambda$ and λ measures the market price for risk.[10]

Figure 8 compares the term structure of futures prices in the random walk model against the term structure for the one-factor reverting model when r=6%, μ=3%, σ=33%, κ=0.3, λ=0.2, P_0^*=20, and P_0 =22.5. In the random walk model this implies that δ=5%. The futures price for the random walk model simply grows at the rate of 3% as specified by the drift parameter and starting from the initial spot price of 22.5. In the reverting model the steady-state price grows at 3%, but starts at a point slightly below the initial spot price, i.e., at 20. The futures price in the reverting model initially declines, reflecting the short-term expectation that the spot price will decline as it approaches the steady-state. The futures price eventually increases, reflecting the long-term expectation that the spot price will increase together with the steady-state price. Note that in the reverting model the expected rate of growth in the spot price is not constant. This is equivalent to saying that the expected convenience yield is not constant.

Figure 9 below draws out how the risk-adjusted distribution identified by the futures prices can deviate from the actual distribution, and why therefore it is necessary to have an estimate of the market price of risk, λ, if we wish to recover the actual distribution. As the figure shows, the futures price in the reverting model may grow at a higher or lower rate relative to the steady-state price depending upon the market price of risk. Notice that the steady-state or long-run expected price is the same in all three scenarios, but the futures prices vary across the three scenarios, sometimes pointing above and sometimes below the long-run forecasted price. This illustrates that the futures price cannot be simply read as an estimator of the future spot price unless the appropriate adjustment is made. It is worth repeating, however, that for many valuation problems it is not necessary to identify the actual distribution since the futures price summarizes all of the information relevant for valuing an anticipated commodity delivery. This is a key insight that is relied upon heavily in the finance literature.

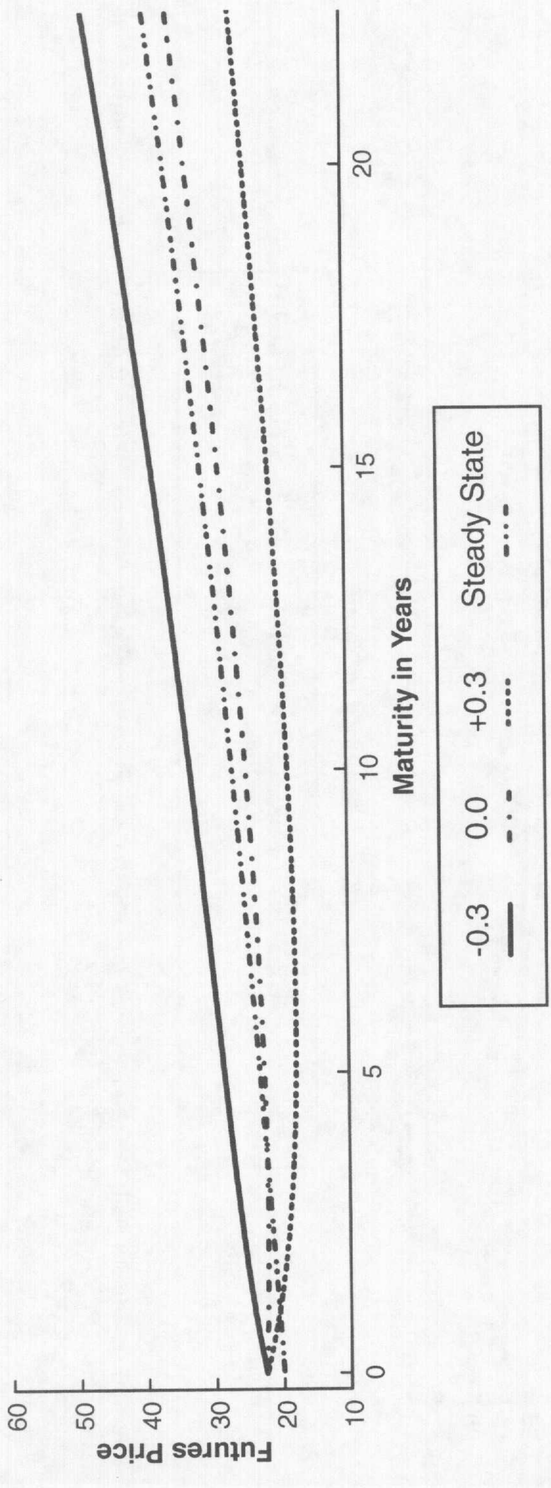
10. The formula below is given in Schwartz (1997) for the case in which μ=0.

Figure 8. A Comparison of the Term Structure of Futures Prices for the Random Walk and Reverting Models



Parameter values for both the random walk and mean reverting are $r=6\%$, $\mu=3\%$, $\sigma=33\%$, $\kappa=0.3$, $\lambda=0.2$, the initial steady state price is $P^* = 20$, and the initial spot price is $P=22.5$.

Figure 9. Value at the End of Delineation



Parameter values for the reverting model are r=6%, μ=3%, σ=33%, κ=0.3, λ=0.2, an initial steady state price of P* =20, and an initial spot price of P=22.5.

Futures Prices in the Two-Factor Reverting Model

The term structure of futures prices in the two-factor model is given by a much more complicated equation than (10) and is given in the appendix. Because the two-factor model is more flexible than either one-factor model, it can generate a wider range of possible term structures of futures prices. This flexibility appears to be necessary in order to fit the historical futures series of many commodities. Schwartz (1997) reports results for three commodities—gold, oil and copper. The value of the flexibility afforded by the two-factor model is illustrated in his results for oil which are reproduced as Figure 10 below. Both the one and the two-factor reverting models are estimated using data from January 1990 through February 1995. Each of the three graphs shows the actual futures price term structure prevailing on a given day together with the predicted term structure from the two models. The figures clearly illustrate the deficiency of the one-factor model compared against the two-factor model.

Despite the initial appearance of complexity in the formula for futures prices in the two-factor model, a basic principle governs. Just as the spot price dynamics of the two-factor model are a blend of the spot price dynamics of the two one-factor models, so too is the term structure of futures prices in the two-factor model a simple blend of the term structures from each of the one-factor models. The reverting process is determinant in the short-run and the random walk process guiding the steady-state is determinant in the long-run.

This means that the dynamics of the steady-state process show up more and more clearly in the dynamics of the longer maturity futures contracts. In the random walk model the futures price is a simple function of the current price and the convenience yield. An analogous relationship holds in the two-factor model. In the long-run, the futures price is determined primarily by the current steady-state price and the steady-state convenience yield:

$$\lim_{T\to\infty} F_t(T) \cong \bar{E}_t(P_T^*) = P_t^* \, e^{(r-\delta^*)(T-t)} \tag{11}$$

where δ^* denotes the unobservable steady-state convenience yield, and where the approximation is due to the overlay of the reverting process. Operationally δ^* is calculated by looking at the convenience yield at a distant horizon, i.e., at $\lim_{T\to\infty}\delta(T)$ where $\delta(T) = \frac{1}{dt} \ln\left[F_t(T + dt) \,/\, F_t(T)\right]$.

Figure 10. Actual Term Structures of Oil Futures and Predicted Term Structures under the One Factor and Two Factor Reverting Models

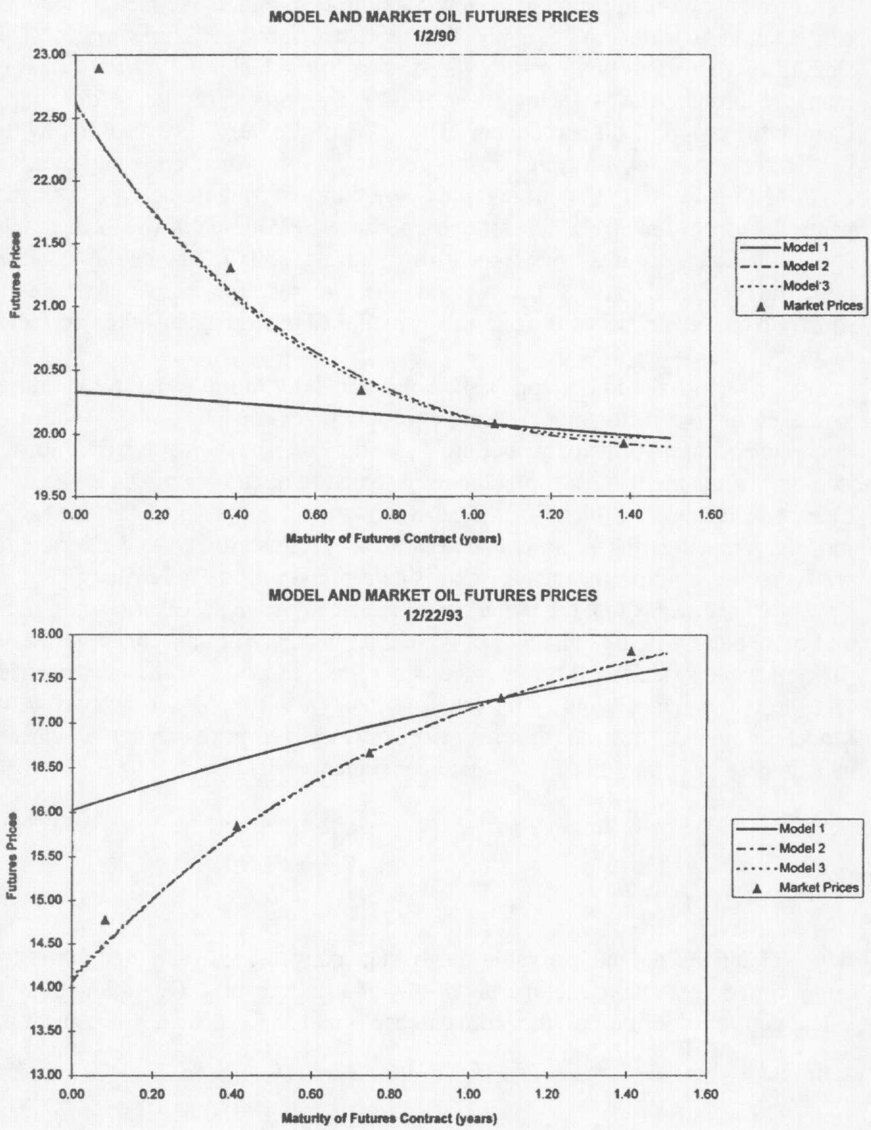
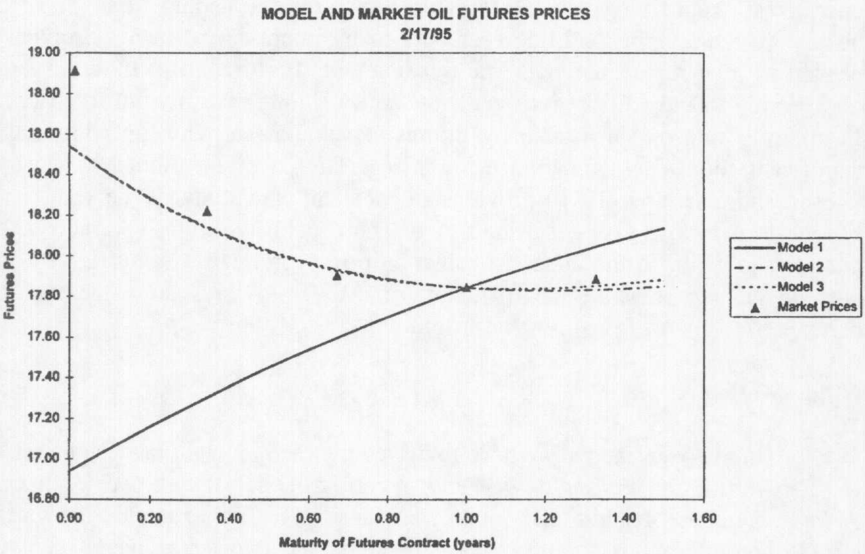


Figure 10. (Continued)



Reprinted from Schwarz (1997), Figure 6.
Model 1 in the figure corresponds to the one factor reverting model discussed in this paper, while Model 2 corresponds to the two factor reverting model. The figure also shows results for Model 3 which is the three factor model discussed later in this paper.

This means that it is possible to construct the estimated series of steady-state prices from a series of long-maturity futures contracts. Using $\lim_{T\to\infty} F_t(T) = P_t^* \, e^{(r-\delta^*)(T-t)}$ from equation (10) above together with the estimated steady-state convenience yield, we have $\hat{P}_t^* = \lim_{T\to\infty} F_t(T) e^{-(r-\delta^*)(T-t)}$. Since the steady-state price is itself a simple random walk, this series can then be used to estimate σ_{P*} directly. Gibson and Schwartz (1991) use this procedure and data covering November 1986 through November 1988 to determine that the volatility on the steady-state price of oil, σ_{P*}, is only 26% although the volatility on the spot price of oil, σ_P, is between 30% and 33%. This method of estimation does not fully exploit the information that can be derived from the full term-structure of futures prices, but it is relatively simple and therein lies its advantage. And although this methodology does not give us any estimates for other key parameters of the two-factor model, we may be able to make good use of these parameters alone since the value of long-lived assets in the oil industry is often dependent more upon the long-run dynamics of the oil price than on the short-run dynamics

Although futures prices do provide important information about the dynamics of commodity prices and the valuation of related capital investments, one caveat should be noted. Observable futures prices seldom exist for time horizons extending beyond one year, and many important valuation problems pertain to long-run valuations. The dynamics of short-maturity futures prices may be significantly different from the dynamics of long-maturity futures prices. In utilizing data on short-maturity futures to value assets with long horizons, significant room for estimation error arises. The paper mentioned above by Gibson and Schwartz (1990) provides an excellent case study of the valuation of long-term oil linked assets using data on short-maturity futures contracts, and attention is given to the apparent valuation errors produced. Schwartz (1997) also documents some of the difficulty of estimating long-run values using parameter estimates from short-term futures.

Other Models

In the pages above we have reviewed the random walk model, the one-factor reverting model, and a two-factor reverting model of commodity prices. A large number of additional models are available. For example, Schwartz (1997) presents a three-factor model based on the two-factor model shown above. The third factor is the level of interest rates. The attractiveness of this model follows from two facts. First, the rate of interest is an important variable in the term structure of futures prices under many models as one can see in equations (8) and (10). Second, the rate of interest varies significantly over time. So much of the variation in the term structure of futures prices at different points in time may reflect variations in interest rates. Incorporating a model of interest rates into a model of commodity prices enables us to extract this important factor and better isolate and estimate the other factors unique to the particular commodity being studied. Schwartz (1997) shows that incorporating the interest rate is important in the case of gold: significant reversion is identified when uncertain interest rates are not incorporated into the model, but disappears when uncertain interest rates are incorporated.

The models presented in this paper are just a starting point. Alternative models can be derived through variation on a number of dimensions. For example, one can construct another two-factor model in which both the price and the volatility parameter itself are uncertain. Or, one can construct other models simply by expanding the number of factors incorporated. This is attractive since the dynamics driving the price of a commodity over time may be very complicated and driven by a large number of factors. As we expand the number of factors incorporated into the model, we increase its flexibility and the range of shapes of the term structure of futures prices that the model can accommodate. Researchers modeling the dynamics of interest rates have developed general models—see for example Heath, Jarrow and Morton

(1992)—and recently these models have been applied to commodity prices—see Cortazar and Schwartz (1994). Of course the additional complexity is purchased at some expense. It is one matter to expand the mathematical power applied to the problem, and quite another to expand our understanding. The simpler models impose a relatively crude structure onto the actual history of commodity prices, and this structure may a priori exclude key aspects of the true dynamics. Nevertheless, the structure imposed is at least one that we understand. The more general models accommodate more complicated structures and allow the data to tell us which structure best captures the true dynamics. However, in many cases we do not have an understanding of the structure produced, so that the technique may purchase little in the way of insight. Moreover, the extra complexity can impose severe burdens on the computations required for many valuations.

CONCLUSIONS

Modern asset pricing methodologies represent an important advance over discounted cash flow valuations. The ability to analyse risk, to value non-linear payoffs efficiently and to incorporate option values are important advantages. These new tools make not only the forecasted level of commodity prices a critical driver of valuations, but also the volatility and dynamic structure of commodity prices, and the pricing of their risk. A model of commodity prices is the engine for these new valuation tools for commodity production projects, and the performance of the tool is in large part dependent of the quality of the engine used. This paper provides an overview of several basic commodity price models that are compatible with modern asset pricing methodologies, and explains the essential differences between the models.

Because modern asset pricing techniques were first developed to value stock price dynamics, the random walk model of stock prices is standard. As these techniques have been applied to valuation of assets contingent on commodity prices, the random walk model has been applied here as well. Unfortunately, the random walk model is ill suited to most commodity markets. A model that recognizes the underlying economics of the commodity market and the equilibrium structure of the commodity price is required. This means that models that incorporate reversion are essential. Nevertheless, in the long-run the commodity price may be well characterized as a random walk. This paper presents a hybrid model that captures both the short-run reversion in commodity prices and the long-run random walk. This model is both simple enough to be used in standard valuation work with some caveats (Laughton 1998b) and complicated enough to capture the actual dynamics of commodity prices. It represents a sound compromise for practical valuation work.

Analyses of commodity prices should incorporate the valuable information provided by futures markets. We have shown how futures prices provide evidence for reversion in certain commodity markets. We have also shown how futures prices can be used to estimate better the parameters of the underlying dynamics of the commodity price. Futures prices provide information on the market value of future cash flows not easily extracted from the spot price series itself.

Appendix: Futures Prices in the Two-Factor Reverting Model

The two-factor reversion model given in equations (5) to (7) was expressed in terms of the unobserved steady state price and the ratio of the current price to the steady state price. No mention was made of the convenience yield. An alternative representation of the two-factor model can be given in terms of the current price and the convenience yield. In the two-factor reversion model the term structure is given by:

$$dP_t = (r-\delta)P_t dt + \sigma_P P_t dz_P \tag{A1}$$

$$d\delta_t = \kappa(\alpha - \delta_t)dt + \sigma_\delta dz_\delta \tag{A2}$$

$$dz_P dz_\delta = \rho_{P\delta} dt \tag{A3}$$

In this case the term structure of futures prices is given by:

$$F_t(T) = P_t \exp[-\delta \frac{1-e^{-\kappa(T-t)}}{\kappa} + A(T)] \tag{A4}$$

where

$$\begin{aligned} A(T) = {} & (r - \hat{\alpha} + \frac{1}{2}\frac{\sigma_\delta^2}{\kappa} - \frac{\sigma_P \sigma_\delta \rho_{P\delta}}{\kappa})(T-t) + \frac{1}{4}\sigma_\delta^2 \frac{1-e^{-2\kappa(T-t)}}{\kappa^3} \\ & + (\hat{\alpha}\kappa + \sigma_P \sigma_\delta \rho_{P\delta} - \frac{\sigma_\delta^2}{\kappa})(\frac{1-e^{-\kappa(T-t)}}{\kappa^2}) \end{aligned} \tag{A5}$$

and where $\hat{\alpha} = \alpha - (\lambda/\kappa)$. Notice that as the volatility of the convenience yield parameter falls to zero, the term structure for the two-factor model reduces to the random walk model with a constant convenience yield. The two factor model in (A1)-(A3) was used in Schwartz (1997). The equivalent representation in terms of the steady state price and the ratio of the current price to the steady state price was derived by Ramzi Zein. Of course equivalence only obtains if the volatility and correlation coefficients are specified appropriately across the two formulations.

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